

# Time and Observables in Unimodular Gravity

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## Abstract

A cosmological time variable is emerged from the hamiltonian formulation of unimodular theory of gravity to measure the evolution of dynamical observables in the theory. A set of 'constants of motion' has been identified for the theory on the null hypersurfaces that its evolution is with respect to the volume clock introduced by the cosmological time variable.

**Keywords:** Relativity, Unimodular, gravity, Time, Observable,

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# 1 Introduction

Research in quantum gravity may be regarded as an attempt to construct a theoretical scheme in which ideas from General Relativity and quantum theory are reconciled. However, after many decades of intense work we are still far from having a complete quantum theory of gravity. Any theoretical scheme of gravity must address a variety of conceptual including the problem of time and identification of dynamical observables. There are many program that attempt to address the above mentioned problems including canonical quantum gravity.

It is well know that some of the issues such as time and observables in quantum gravity have their roots in classical general relativity; in such cases it seems more reasonable to identify and perhaps address the problem first in this context. The classical theory of gravity is invariant under the group of  $\text{Diff}(\mathcal{M})$  of diffeomorphisms of the space-time manifold  $\mathcal{M}$ . This goes against the simple Newtonian picture of the a fixed and absolute time parameter. The classical theory, while itself free from problems relating to the definition and interpretation of time, contains indications of problems in the quantum theory, where the absence of a time parameter is hard to reconcile with our everyday experience. In fact, one can see that in the hamiltonian formulation of classical general relativity, time is suppressed from the theory. There are many proposals for dealing with this question which generally involve a re-interpretation of the usual notion of time ( see [1] for an overview of these proposals).

Unimodular gravity as an alternative theory of gravity was originally considered by Einstein [2] cast into canonical form by Unruh [3] and others [4] for the purpose of constructing an explicit time variable for the theory. It contains a pair of canonically conjugate fields that are not present in the canonical formulation of conventional General Relativity. One of the new fields specifies the value of the cosmological constant, while the conjugate field carries the information

about the space-time volume bounded by the initial and final space-like hypersurfaces. The four-volume variable may be regarded as a cosmological time. In fact, in formulating the Einstein theory of relativity, one chooses to limit the geometries by specifying a fixed value for the total four-volume. This produces unimodular theory whose classical limit is equivalent to the Einstein theory except that the cosmological constant becomes a constant of integration, rather than a dynamically unchangeable parameter in the Lagrangian. Limiting the geometries in this way may solve the timeless character of the quantum gravity.

Identification of dynamical observable for the theory is another fundamental issue that has its roots in classical formulation of general relativity and directly related to the issue of time. The problem of evolving of a dynamical system from initial data is known as the Cauchy problem or initial value problem [5] and in General Relativity is naturally addressed using the 3+1 ADM representation. In the ADM approach, the spatial hypersurface  $\Sigma$  is assumed to be equipped with a space-like 3-metric  $h_{ij}$  induced from space-time metric  $g_{\mu\nu}$ . Einstein's equations are of course covariant and do not single out a preferred time with which to parametrise the evolution. Nevertheless, we can specify initial data on a chosen spatial hypersurface  $\Sigma$ , and if  $\Sigma$  is Cauchy, we can evolve uniquely from it to a hypersurface in the future or past. The issue of specification of initial or final data on Cauchy hypersurfaces has been discussed in many papers; for example, see [6].

An alternative approach to Cauchy problem is known as characteristic initial value problem in which one may fix the initial data on null hypersurfaces rather than spatial hypersurfaces. There are reasons to motivate us using null boundaries in formulating general relativity. First, the procedure of determining 'initial conditions' on space-like hypersurface is unrealistic and unnatural in the context of relativity [7]. This is because no information can be obtained from space-time points which are separated by space-like distances. In particular, an observer

has access only to information originated from his past light cone [8]. This is an immediate consequence of the laws of relativity, if we assume that physical observations are made by a single localized observer [9]. Second, there has been considerable success in using null boundaries to formulate the canonical theory of gravitational radiation on outgoing null surfaces. This is because in electromagnetism and gravitation (which are mediated by particles with zero mass), fields propagate in null directions and along null hypersurfaces [10]. Third, in some cosmological models of interest, space-time is not globally hyperbolic and so there are no Cauchy hypersurfaces on which to specify boundary data. In such cases, data specified on a space-like hypersurface cannot be used to generate a unique classical solution and therefore cannot be used to label a particular point in the phase space. Even if the space-time is globally hyperbolic, it may not be possible for localized observers to gather all the necessary boundary data from a space-like Cauchy hypersurface. Indeed, unless the space-time is deterministic, there will be no event whose casual past contains the hypersurface. In this case, no localized observer will have access to enough data to distinguish between different classical solutions - i.e. between different elements of the phase space. Forth, the formulation of gravitational radiation field on the null surface lays bare the dynamical degrees of freedom in the theory and allows one to analyze the properties of the gravitational radiation field in terms of these quantities [11][12][13].

In addition, the approach of setting the final data on a null hypersurface is essential if we are interested in a theory such as quantum theory that observations made by a single localized observer who can collect observational data only from that subset of space-time which lies in the causal past[9].

In this paper in section two, the hamiltonian formulation of unimodular gravity is developed. As a product a time variable has been emerged from the theory that can be regarded as a cosmological time

variable.

In section three, a discussion of Dirac observables in general relativity is given. In addition, Rovelli's constants of motion [17] have been introduced. Section four introduces a set of observables for the theory on the past light cone of a single localized observer. These observables are similar to Rovelli's constants of motion on null hypersurfaces. The evolution of these observables is with respect to time variable obtained from unimodular theory of gravity.

## 2 The unimodular gravity and emerge of time

The Einstein-Hilbert action for General Relativity is given by

$$S[g_{\mu\nu}] = \int |det(g_{\mu\nu})|^{1/2} R[g_{\mu\nu}] d^4x. \quad (1)$$

where  $R$  be the Riemann scalar computed from the metric tensor  $g_{\mu\nu}$ .

The equations of motion for unimodular gravity can be obtained by varying the Hilbert action (1) subject to the unimodular coordinate condition,

$$- |det(g_{\mu\nu})|^{1/2} + 1 = 0. \quad (2)$$

The theory is then equivalent to General Relativity with an unspecified cosmological constant, the latter appearing as a dynamical variable unrelated to any parameters in the action.

An alternative way to obtain the same theory is to include an extra term in the Hilbert action, so that the new action is

$$S[g_{\mu\nu}] = \int e [R - \frac{1}{8} (\nabla_\mu M^\mu)^2] d^4x, \quad (3)$$

where  $e = |det(g_{\mu\nu})|^{1/2}$ . The field equation for  $M^\mu$  gives rise to an unspecified cosmological constant in the action.

In the present paper, our considerations will be based on a Lagrangian formulation for its relative simplicity. Nevertheless, we are interested to sketch here how the unimodular assumption manifests itself in Hamiltonian versions of gravity.

For this purpose, we now consider the Hamiltonian formulation of the theory. We rewrite the action as

$$S = \int [eR - \frac{1}{8}e^{-1}(\partial_\mu m^\mu)^2] d^4x \quad (4)$$

where  $m^\mu = eM^\mu$  and  $\partial_\mu m^\mu = \dot{m} + \partial_i m^i$ .

The momentum associated with the dynamical variable  $m^0(x)$  is then

$$\pi_0(x) = -\frac{1}{4}(\nabla_\mu M^\mu)|_x, \quad (5)$$

while the momentum associated with the dynamical variables  $m^i(x)$  are

$$\pi_i(x) = 0. \quad (6)$$

Since the action does not explicitly depend on the variables  $\dot{m}^i$ , the vanishing momenta  $\pi_i$  are primary constraints.

$$\pi_i(x) \approx 0. \quad (7)$$

To ensure that these primary constraints are preserved with time evolution, we also require that  $\dot{\pi}_i(x) \approx 0$ , which implies that

$$\partial_i \pi_0|_x \approx 0. \quad (8)$$

The secondary constraints, eq (8), ensure that  $\pi_0(x)$  is spatially constant. By substituting  $\pi_0(x)$  back into the action one obtains

$$S = \int e[R - 2\pi_0^2] d^4x. \quad (9)$$

One recognizes that  $\pi_0(x)$  is simply the square root of the cosmological constant.

The Hamiltonian is obtained from the Lagrangian by a Legendre transformation and is given by

$$H = \int_{\Sigma} [N(\mathcal{H} - 2\sqrt{{}^{(3)}h} \pi_0^2) + N^i \mathcal{H}_i + \lambda^i \pi_i - \pi_0(\partial_i m^i)] d^3x, \quad (10)$$

where  $\sqrt{{}^{(3)}h} d^3x$  is the measure associated with the 3-metric  $h_{ij}$  on  $\Sigma$ . The three-dimensional manifold  $\Sigma$  is a submanifold of the space-time  $\mathcal{M}$ . The variables  $N, N^i, \lambda^i$  and  $m^i$  in Hamiltonian equation are Lagrange multipliers and  $\mathcal{H}$  and  $\mathcal{H}_i$  are the Hamiltonian and momentum given by

$$\mathcal{H}_i(x; h_{ij}, \pi^{ij}) := -2\pi_{i|j}{}^j(x) \quad (11)$$

and

$$\mathcal{H}(x; h_{ij}, \pi^{ij}) := \mathcal{G}_{ijkl}(x, h_{ij}) \pi^{ij}(x) \pi^{kl}(x) - |h|^{\frac{1}{2}}(x) R(x, h_{ij}) \quad (12)$$

in which

$$\mathcal{G}_{ijkl}(x, h_{ij}) := \frac{1}{2} |h|^{1/2}(x) [h_{ik}(x) h_{jl}(x) + h_{jk}(x) h_{il}(x) - h_{ij}(x) h_{kl}(x)] \quad (13)$$

We thus have a new Hamiltonian constraint

$$\mathcal{H}_1 = \mathcal{H} - 2\sqrt{{}^{(3)}h} \pi_0^2 \approx 0, \quad (14)$$

instead of  $\mathcal{H} \approx 0$ .

It can be shown that the new Hamiltonian constraint, the momentum constraints and the constraints (5) and (7) are all first class.

It has been shown by Henneaux, Teitelboim [4] and separately by Unruh [3] that the cosmological constant may be regarded as the momentum conjugate to a dynamical variable which may be interpreted as the cosmological time parameter ,

$$T(t) = \int_{\Sigma_t} n_{\mu} M^{\mu} \sqrt{{}^{(3)}h} d^3x, \quad (15)$$

with  $n_\mu$  and  $\sqrt{{}^{(3)}h}$  are respectively unit normal to  $\Sigma_t$  and the square root of determinant of the 3-metric on  $\Sigma_t$ . An application of Stokes' theorem shows that  $T(t)$  is invariant under  $\delta M^\mu = \epsilon^{\mu\nu\rho\sigma}\nabla_\nu N_{\rho\sigma}$ , as it should be.

The equation of motion for  $T(t)$  derived from (15),

$$\frac{dT}{dt} = \int_{\Sigma_t} N \sqrt{{}^{(3)}h} d^3x = \int_{\Sigma_t} \sqrt{{}^{(4)}g} d^3x, \quad (16)$$

implies that  $T(t)$  is just the 4-volume preceding  $\Sigma_t$  plus some constant of integration. Integration with respect to  $t$ , this means that, the change of the time variable equals the four-volume enclosed between the initial and final hypersurfaces, which is necessarily positive. This time variable,  $T(t)$  may be regarded as a cosmological time variable, as it continuously increases along any future directed time-like curve [15]. Therefore one may consider  $T$  as a monotonically increasing function along any classical trajectory and so can indeed be used to parametrise this trajectory.

### 3 Dirac observables in General Relativity

General Relativity, like many other field theories, is invariant with respect to a group of local symmetry transformations [16]. The local symmetry group in General Relativity is the group  $\text{Diff}(\mathcal{M})$  of diffeomorphisms of the space-time manifold  $\mathcal{M}$ .

In General Relativity, Dirac observables [14] must be invariant under the group of local symmetry transformations. The Hamiltonian constraint and momentum constraint in General Relativity are generators of the symmetry transformations, and so a function  $\Phi$  on the phase space is a Dirac observable, *iff*

$$\{\Phi, \mathcal{H}\} = \{\Phi, \mathcal{H}_i\} = 0, \quad (17)$$



at all points  $x \in \mathcal{M}$ . Such observables are necessarily constants of motion. They are invariant under local Lorentz rotations  $SO(3)$  and  $Diff\Sigma$  (as well as  $SO(1, 3)$ ).

The above criteria for observables in relativity appear to rule out the existence of local observables if locations are specified in terms of a particular coordinate system. Indeed, it might appear that one would be left with only observables of the form

$$\Phi = \int \phi(x) \sqrt{-g(x)} d^4x, \quad (18)$$

where  $\phi(x)$  is an invariant scalar as for example  $R$ ,  $R^2$ ,  $R^{\mu\nu}R_{\mu\nu}$ . While such observables clearly have vanishing Poisson brackets with all the constraints, they can not be evaluated without full knowledge of the future and past of the universe. While this may be deducible in principle from physical measurements made at a specific time, it is well beyond the scope of any real experimenter.

However, in reality, observations are made locally. We therefore ought to be able to find a satisfactory way to accommodate local observables within General Relativity. In particular, we would like to be able to talk about observables measured at a particular time, so that we can discuss their evolution. Local observables in classical or quantum gravity must be invariant under coordinate transformations. The difficulty in defining local observables in classical gravity is that diffeomorphism invariance makes it difficult to identify individual points of the space-time manifold [18].

It is fairly easy to construct observables which commute with the momentum constraints. Such observables can be expressed as functions of dynamical variables on the spatial hypersurfaces. However, according to the Dirac prescription, observables must also commute with Hamiltonian constraint.

In a slightly different formalism, Rovelli addressed the problem by introducing a Material Reference System (*MRS*) [17]. By *MRS*, Rovelli

means an ensemble of physical bodies, dynamically coupled to General Relativity that can be used to identify the space-time points.

In Rovelli's approach, all the frames and all the test particles are assumed to be material objects. However, to implement the process and simplify the calculation, one has to neglect the energy-momentum tensor of matter fields in the Einstein equations, as well as their contributions to the dynamical equations for matter fields [19]. Of course the price that one has to pay for this neglect is obtaining an indeterministic interpretation of the Einstein equations. General Relativity is then approximate because we disregard the energy-momentum of the *MRS* as well as incomplete (because we disregard dynamical equations of the *MRS* [20]). However, the indeterminism here is not fundamental and does not imply that Dirac determinism is violated [17]. In fact this approximation can arise in any field theory and has always been resolved by considering a limiting procedure in which the rest masses, charges, etc., of test bodies tends to zero [21].

Rovelli's observables can be interpreted as the values of a quantity at the point where the particle is and at the moment in which the clock displays the value  $t$ . However  $t$  itself is not an observable, even though its conjugate momentum is constant along each classical trajectory.

By introducing a cloud of particles filling space, with a clock attached to every particle, one can easily generalize the model to a continuum of reference system particles, in order to get a complete material coordinate system and a complete set of physical observables. Rovelli's 'evolving constants of motion' are genuine Dirac's observables. They are constant of motion since they commute with Hamiltonian and momentum constraints, while evolving with respect to the clock time  $t$ .

Rovelli's observables are functions defined on spatial hypersurfaces. He assumes the space-time has a topology  $\Sigma \times R$  where  $\Sigma$  is a compact spatial hypersurface and  $R$  is the real time. In order to have evolution into the future or past the spatial hypersurface must be a Cauchy

hypersurface. This makes sense if the underlying space-time is assumed to be globally hyperbolic.

Perhaps more importantly, the observations collected by the observers will not generally be accessible to any single observer, and so Rovelli's approach is not useful if we set a theory of observations by a single observer.

As discussed, one may fix the initial data on null hypersurfaces rather than spatial hypersurfaces. In General Relativity it is natural to work with a foliation of space-time by space-like hypersurfaces, as this reflects the older Newtonian idea of a 3-dimensional universe developing with time. This seems close to our experiences and is easy to visualize. Nevertheless, null hypersurfaces and null directions should be considered in here for the reasons already discussed in the introduction. In particular The approach of setting the final data on a null hypersurface is essential if we are interested in a theory such as quantum theory that observations made by a single localized observer who can collect observational data only from that subset of space-time which lies in the causal past.

## 4 Constants of motion

In ADM formalism, the space-time  $\mathcal{M}$  is assumed to be foliated by a coordinate time  $t$ . Now, suppose that we choose the foliated 3-geometry,  $\Sigma(t)$  to be observer's past light-cone and also the space-time contains a future-directed time-like geodesic  $\Gamma$  representing the world-line of an observer. Also suppose that the 4-volume time variable  $T(t)$  defined in (15) instead of coordinate time  $t$  has been used to label the 3-surfaces and also the future-directed time-like geodesic  $\Gamma$ . We also suppose that the metric  $g$  satisfies unimodular Einstein's equations which are assumed to include a contribution from the cosmological constant. It is then possible to construct a covariantly defined quantity determined

by field values on  $\Sigma_T(t)$

$$\Phi(\Sigma_T) = \int_{\Sigma_T} \phi(x) \sqrt{{}^{(3)}h(x)} d^3x, \quad (19)$$

where  $\phi(x)$  is any scalar invariant on  $\Sigma_T(t)$  expressible in terms of  $h_{ij}$ ,  $R^i_{jkl}$ ,  $K_{ij}$ . ( $i, j, k, l$  are spatial indices running from 1 to 3) and their covariant spatial derivatives. These quantities are called world line  $\Gamma$ -observables [15].

The so called  $\Gamma$ -observables then have vanishing poisson brackets with any Hamiltonian  $H$ , equation (10), which generates time translations of  $\Sigma_T(t)$  along  $\Gamma$ . The observables  $\Phi(\Sigma_T)$  have vanishing Poisson brackets with the momentum constraints since they are covariantly defined functions of the variables on the 3-surfaces  $\Sigma_T(t)$ . However, they do not have vanishing Poisson brackets with the Hamiltonian constraints  $\mathcal{H}_1$ , since the prespecified foliation is not invariant under local time evolution [22].

If we define new quantities,  $\Phi_T(\Sigma_T)$ ; the value  $\Phi(\Sigma_T)$  at a certain time  $T$ , then these quantities have vanishing Poisson brackets with the integrated Hamiltonian constraints,  $\{\Phi_T(\Sigma_T), \int \mathcal{H}_1 d^3x\} = 0$ , and can be called 'evolving constants of motion'. These observables are not the same as Rovelli's constants of motion in a sense that they are not genuine Dirac's observables. Similarly, the dynamical time  $T(t)$  in the new labeling of 3-surfaces is not a Dirac observable. The evolution of these observables is expressed in terms of the dynamical variable  $T$ , whose conjugate momenta,  $\pi_0$  is a first class constraint.

In summary we have seen that an explicit time variable has been emerged from unimodular theory of gravity, interpreted as a cosmological time, and can be used by observers as a clock to measure the passage of time. A set of 'evolving constant of motion' has been constructed by using the dynamical time variable emerged from unimodular gravity which set the condition on the  $\Gamma$ -observables.

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## References

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